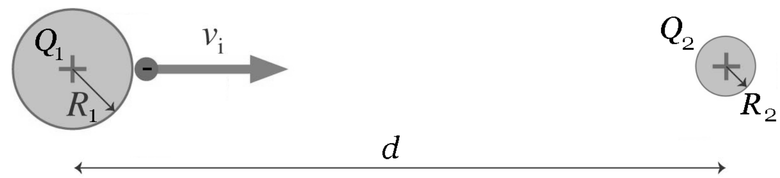
PHYS 2212 **Solution**
Recitation 04 February 15–18

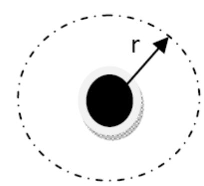
Example

A uniformly charged sphere of radius R_1 bears a charge Q_1 and is centered at the origin. Another uniformly charged sphere with radius $R_2 < R_1$ and charge $Q_2 = Q_1/4$ is centered at $x = d$ on the x axis ($d \gg R_1 + R_2$). What is the minimum speed with which an electron projected from the surface of the first sphere along the positive x axis reaches the second sphere?



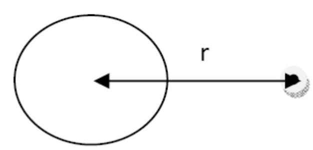
The electric field due to a charged sphere outside the sphere can be calculated using Gauss' Law. Consider a spherical Gaussian surface, concentric with the charged sphere, of radius r>R (R is the radius of the charged sphere) .From Gauss' law

$$\oint_{GSurface} \overline{E}.\overline{dA} = \frac{q_{enclosed}}{\varepsilon_0} \quad => \quad 4\pi r^2 E(r) = \frac{q_{sphere}}{\varepsilon_0} \quad => \quad E(r) = \frac{q_{sphere}}{4\pi\varepsilon_0 r^2}$$

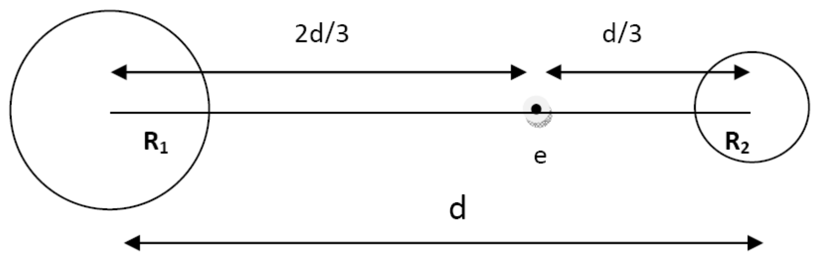


Thus, the electric field outside a charged sphere is the same as that due to a point charge of equal charge placed at the center of the sphere. Hence the potential energy between a charged sphere and the electron is the same as that of the point charge replacing the sphere (and placed at its center) and the electron.

$$-\frac{kQe}{r} = U$$



When the electron is projected from the surface it experiences a pull towards origin due to the electric field of sphere R_1, Q_1. Simultaneously, it experiences a pull by R_2, Q_2. Since the force due to R_1, Q_1 is greater near the surface of the sphere R1,Q1 the NET FORCE on electron is towards origin. Thus, the electron slows down as it moves towards R_2,Q_2.



There exists a point BETWEEN THE SPHERES where the fields due to both the spheres cancel each other (it is not necessary that such a point always exists. But given that d >> R_1+R_2 and $Q_2 = Q_1/4$ it does exist). Beyond this point the force on the electron is towards the sphere R_2,Q_2. Thus we project the electron such that it JUST manages (means final speed at the equilibrium point is zero) to reach this "Equilibrium Point". Beyond this point, the electron moves due to the greater pull from R_2,Q_2 and reaches the sphere R_2,Q_2 .

Finding the Equilibrium Point: Let the equilibrium point be at a distance r_1 from origin and r_2 from X=d on the X-Axis. $r_1+r_2= d$

$$\frac{kQ_1}{r_1^2} = \frac{kQ_2}{r_2^2} = \frac{kQ_1}{4r_2^2} \Rightarrow 2r_2 = r_1$$

Solving for r_1 and r_2 we find that $r_1 = 2d/3$ and $r_2 = d/3$

Energy Conservation: The velocity of the electron must be zero when it reaches this equilibrium point. The Sum Total of the Kinetic and Potential Energy of the system is conserved

$$K_{initial} + U_{initial} = K_{final} + U_{final}$$

$$U_{initial} = -\frac{kQ_1e}{R_1} - \frac{kQ_2e}{d-R_1} = -\frac{kQ_1e}{R_1} - \frac{k\frac{Q_1}{4}e}{d-R_1} = -\frac{kQ_1e}{R_1} - \frac{kQ_1e}{4(d-R_1)} \qquad K_{initial} = \frac{1}{2}m_ev^2$$

$$U_{final} = -\frac{kQ_1e}{r_1} - \frac{kQ_2e}{r_2} = -\frac{kQ_1e}{\left(\frac{2d}{3}\right)} - \frac{kQ_2e}{\left(\frac{d}{3}\right)} = -\frac{kQ_1e}{\left(\frac{2d}{3}\right)} - \frac{k\frac{Q_1}{4}e}{\left(\frac{d}{3}\right)} = -\frac{9kQ_1e}{4d} \qquad K_{final} = 0$$

$$-\frac{kQ_1e}{R_1} - \frac{kQ_1e}{4(d-R_1)} + \frac{1}{2}m_ev^2 = -\frac{9kQ_1e}{4d} \qquad (K_{initial} + U_{initial} = K_{final} + U_{final})$$

$$\frac{1}{2}m_ev^2 = \frac{kQ_1e}{R_1} + \frac{kQ_1e}{4(d-R_1)} - \frac{9kQ_1e}{4d} \Rightarrow v^2 = \left(\frac{2kQ_1e}{m_e}\right)\left(\frac{1}{R_1} + \frac{1}{4(d-R_1)} - \frac{9}{4d}\right)$$

$$v = \sqrt{\left(\frac{kQ_1e}{2m_e}\right)\left(\frac{4d^2 - 12R_1d + 9R_1^2}{R_1d(d-R_1)}\right)}$$

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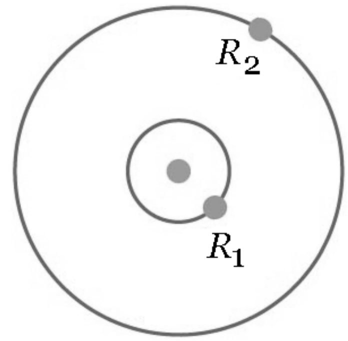
Problem

A simple model of an atom with one electron suggests that the electron (charge $-e$, mass m) moves in circular orbits under the Coulomb force of attraction around a stationary nucleus (charge $+Z$, mass $M \gg m$). What is the atom's energy change when the electron changes from an orbit of radius R_1 to radius $R_2 > R_1$?.



The Coulomb force of attraction between a positively charged nucleus (charge **Ze**) and the negatively charged electron (charge **–e**) provides the centripetal force for the electron to move in a circular orbit. Assume that the speed of the electron in the orbit of radius r is "v". Equating $m_e a_{centripetal}$ and $F_{coulomb}$ we get

$$m_e \frac{V^2}{r} = \frac{k(Ze)e}{r^2} => m_e v^2 = \frac{kZe^2}{r} \qquad (1)$$

(m_e is the mass of the electron) . The potential energy of the Nucleus – Electron combination is given by

$$U = \frac{-k(Ze)e}{r} = \frac{-kZe^2}{r} \qquad (2)$$

The Kinetic Energy of the electron in the orbit is given by

$$K = \frac{1}{2} m_e v^2 = \frac{1}{2}\frac{kZe^2}{r} \qquad \text{from (1)} \qquad (3)$$

Thus the Total Energy (E) of the Nucleus-Electron system is

$$E = U + K = \frac{-kZe^2}{r} + \frac{1}{2}\frac{kZe^2}{r} = \frac{1}{2}\frac{kZe^2}{r}$$

$$E = -\frac{1}{2}\frac{kZe^2}{r} \qquad (4)$$

Observe that the energy of the atom is "Negative", implying that work needs to be done in separating the electron from nucleus. Also, a very puzzling aspect is that the mass of the nucleus and that of the electron do not figure into this expression. The assumption M>>>m makes it possible to keep the nucleus stationary. However, the nucleus being very very heavy wobbles "minutely" as the electron moves around it - since the nucleus too experiences a pull from electron. Even without assuming that the nucleus is stationary we can solve the problem yielding the same energy in Center of Mass Frame.

As an electron jumps from an orbit of radius R_1 to another of radius R_2 the change in energy is

$$\Delta E = E_f - E_i = -\frac{1}{2}\frac{kZe^2}{R_2} + \frac{1}{2}\frac{kZe^2}{R_1} = \frac{kZe^2}{2}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \qquad (5)$$